by Paul Ellerman,
Director of Reliability
pellerman@microsemi.com

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Calculating Chi-squared (X^2) for Reliability Equations

Overview

Establish an accurate method for calculating the value of Chi-squared (X^2) used in generating reliability values such as *Failure Rate* (λ), *Failures in Time* (FIT) and *Mean Time to Failure* (MTTF) without using the traditional, out-dated practice of looking up the value in a published table which may or may not have the specific values desired. The focus of this paper is to present *what* the predominant methods are for calculating X^2, *why* one method is preferred over another and ultimately *how* to accurately perform the calculation. The goal of this paper is not meant to provide extensive insight into the proofs and inner workings of statistics. This paper is a reference text for *Calculating Reliability using FIT & MTTF: Arrhenius HTOL Method* also by this author.

Chi-Square (X^2)

$$X_\alpha^2 \ \text{ or } \ X^2(\alpha, \nu)$$

where: α **(alpha)**, *confidence level* (CL) or *probability*, is the applicable percent area under the X^2 probability distribution curve; reliability calculations use $\alpha = 0.6$ (or 60%). Note: The *total area* under the X^2 curve is always 1, so $\alpha \leq 1$ (or 100%).

ν **(nu)**, *degrees of freedom* (DF), determines the shape of the X^2 curve; reliability calculations use $\nu = 2r + 2$ where r = number of failures or rejects.

X^2 Approximation Equation

The most accurate of several X^2 *Approximation Equations* used over the years for calculations involving *large degrees of freedom* (ν) is:

$$X_\alpha^2 = \nu \left(1 - \frac{2}{9\nu} + z_\alpha \sqrt{\frac{2}{9\nu}} \right)^3$$

As with all reliability calculations of X^2, we convert the *degrees of freedom* (ν) to *number of failures* (r) using the equation $\nu = 2r + 2$:

$$X_\alpha^2 = 2(r+1) \left(1 - \frac{1}{9(r+1)} + z_\alpha \sqrt{\frac{1}{9(r+1)}} \right)^3$$

The remaining variable z_α is determined by using another common probability curve, the *Student's t-Distribution*, for a specific *confidence level* (CL) and *degrees of freedom* (DF) = ∞.

CL DF	0.6	0.75	0.9	0.95	0.975	0.99	0.995	0.9995
∞	0.253	0.674	1.282	1.645	1.960	2.326	2.576	3.291

If we choose CL of 0.6 or 60% to determine the z_α variable, just as we normally do for reliability calculations, the "X^2 Approximation Equation" becomes:

$$X_\alpha^2 = 2(r+1)\left(1 - \frac{1}{9(r+1)} + 0.253\sqrt{\frac{1}{9(r+1)}}\right)^3$$

Note: Historically this specific formula has been used at several Microsemi (MSCC) facilities as the core formulary for reliability calculations.

As described in several texts cited as reference material for this paper, the above equation renders a good approximation of X^2 for $v > 30$ (or r > 14). In actual application, this equation works well for $v > 2$ (or r > 0), but produces a significant error for $v = 2$ (or r = 0). The table below shows the decimal place accuracy of this equation through $v = 42$ (or r = 20):

X^2 Approximation Equation Accuracy

v (DF)	# of Failures (r)	Decimal Place Accuracy
2	0	1
4 thru 42	1 thru 20	2

Note that there are also significant errors produced for smaller values of $v < 2$, but those values do not occur in this application since there cannot be fewer than r = 0 failures.

CHIINV Function in MS Excel

In MS Excel 2003 and later versions, the CHIINV function calculates X^2 from the input data *probability* and *deg_freedom* rendering accurate values throughout the entire use range up to 4-decimal places. This method is currently used as a standard in the statistics community for calculating values for the X^2 distribution. In Excel, the formula is expressed as:

=CHIINV(probability,deg_freedom)

Determining v **(nu),** *degrees of freedom* or *deg_freedom*, we use the equation described previously where *r = number of failures or rejects*:

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$$v = 2(r+1) = 2r + 2$$

Since reliability calculations require the *left* portion of area under the X^2 distribution curve and CHIINV happens to calculate the *right* side, we must use the equation below to determine the correct value for *probability*:

$$CHIINV(probability) = 1 - \alpha$$

when using decimal values for α, or

$$CHIINV(probability) = 1 - \alpha/100$$

when using percentage values for α.

Example: for α = .60 or 60%, CHIINV(probability) = .40 or 40% respectively.

NOTE: For a more detailed discussion on the subject of the *left vs. right* portion of the area under the X^2 probability curve, refer to Appendix A at the end of this article.

Conclusion

The conclusion is that the X^2 *Approximation Equation* is, for the most part, accurate enough to calculate FIT and MTTF for n > 2 or r > 0, however with the emergence of improved statistical functions within MS Excel such as CHIINV, it is much simpler as well as more accurate to use those imbedded functions especially when building a user friendly calculator in MS Excel or MS Access.

Reference Material

1. Samuel M. Selby, Ph.D. Sc.D., *Standard Mathematical Tables*, 12th ed., pg. 617-619, The Chemical Rubber Co., Cleveland, Ohio, 1972.
2. Mary Gibbons Natrella, *Experimental Statistics*, National Bureau of Standards Handbook 91, pg. T-4, United States Department of Commerce, Washington D.C., 1963.
3. Thomas Pyzdek, *The Six Sigma Handbook*, Appendix Table 4, pg. 735-737, McGraw-Hill, New York, 2003
4. Patrick D. T. O'Connor, *Practical Reliability Engineering*, 4th ed., John Wiley & Sons, UK, (2010)
5. Texas A & M Website, *X^2 Internet Calculator*, http://www.stat.tamu.edu/~west/applets/chisqdemo.html, one of several online X^2 calculators.
6. QuickLogic Website Article, *QuickLogic Reliability Monitor*, http://klabs.org/richcontent/Reports/Q_Logic_RELMON98.PDF
7. Vicor Website Article by Scott Speaks, *Reliability and MTBF Overview,* http://www.vicorpower.com/documents/quality/Rel_MTBF.pdf

Appendix A
X^2: Making sure you know which side you're on.

In researching published tables for determining X^2 values, it became evident that there was, at some point in the past, a shift in perspective for the majority of the tables made available. In two of the references used for this article, specifically Ref 1: Selby, *Standard Mathematical Tables* and Ref 2: Natrella, *Experimental Statistics*, the X^2 value was calculated using the percent area under the distribution curve (α : confidence level), from the *lower limit* or the *area to the left* (Figure 1). Note that both of these texts refer to even older tables going back to 1938.
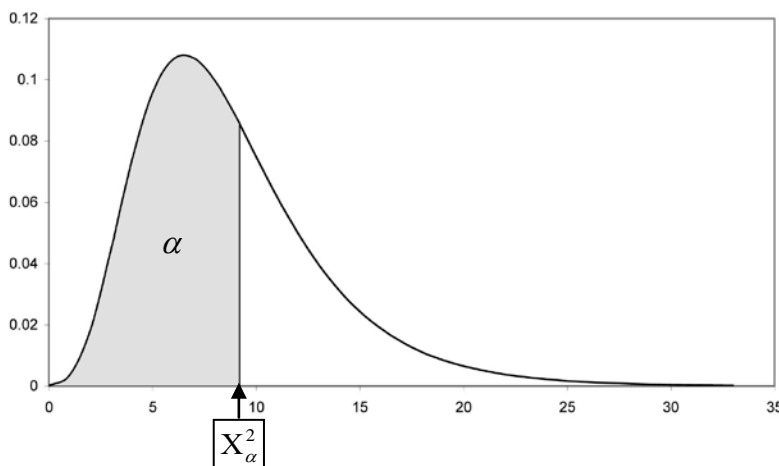

Figure 1: X^2 value calculated from the *lower limit* or the (α) *area to the left*

It should also be mentioned that the various X^2 *Approximation Equations* shown in both Selby and Natrella also calculate X^2 from the *lower limit*.

More recently, the focus of X^2 calculations shifted from using the *lower limit* calculation to the *upper limit* or the *area to the right* (Figure 2).
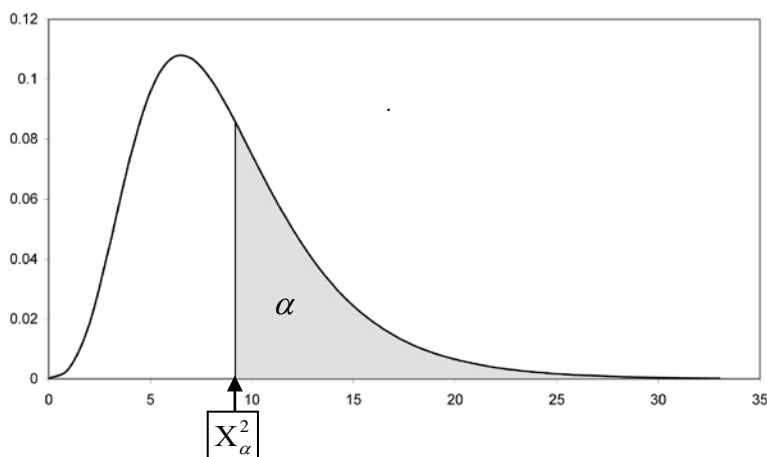

Figure 2: X^2 value calculated from the *upper limit* or the (α) *area to the right*

Some examples of solving for the *upper limit* are found in Ref 3: Pyzdek, *The Six Sigma Handbook*, 2003 and Ref 5: Texas A & M Website, *X^2 Internet Calculator* as well as the CHIINV function found in Excel. Therefore, when using such tables and calculators to determine X^2 for reliability calculations, the (α) *confidence level* or *probability* must be converted using the previously mentioned equation:

$$calculation\_probability = 1 - \alpha$$

Examples of this conversion can be found in the QuickLogic Website Article, *QuickLogic Reliability Monitor*, and the Vicor Website Article by Scott Speaks, *Reliability and MTBF Overview*.

Many but not all tables and calculators indicate which side of the distribution is being used. In any case one must be clear which side is being used and which side is required. Since reliability calculations use the *probability of failure* to statistically determine the *Failure Rate* rather than the *probability of survival*, caution must be taken to assure the calculation of X^2 uses the *lower limit*. A prime example of a modern text where the *lower limit* is used appears in the tables found in Ref 4: Patrick D. T. O'Connor, *Practical Reliability Engineering*, 2010.

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